Beta Groups

Algirdas Javtokas

Department of Mathematics and Informatics, Vilnius University
Naugarduko 24, 03225 Vilnius, Lithuania
ajavtokas@math.com

Abstract

Paper introduces algebraic structure with infinitely many identity elements and investigates its primal properties, some of which coincides with properties of groups.

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1 Introduction

In group theory axiom system is chosen so, that we can easily prove uniqueness of identity or inverse element. But naturally emerges question is it possible to define consistent mathematical structure with infinitely many non trivial identities? And is this new structure maintains group properties? We want to begin a discussion about possible realizations of such structures. This paper suggest axiom system, which is consistent and in some sense it generalizes a group concept. We can identify some analogy with a group of units in associative ring, where some recent developments (see Bovdi, 2005; Coleman and Easdown, 2000; Dekimpe, 2003; Giambruno, Jespers and Valenti, 1994; Giambruno, 1996; Hill, 1994; Jespers and Polcino, 1996; Kawai, 2005; Li and Parmenter, 2005; Pita, del Río and Ruiz, 2005; Szechtman, 2004; Wilcox, 2004) covers broad range of topics, but our concept of many "units" is lying in the core of chosen axiomatic system, it is not definable in secondary steps. The most surprising fact about new structure is that it maintains some of classical groups properties.

2 Beta groups

Let B be a set and $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ its subsets which we call a subset of identity elements, inverse elements and the rest elements respectively, and $B = B^{(1)} \cup B^{(2)} \cup B^{(3)}$.

Definition 2.1. *A β-group $(B, *)$ is a set B together with a binary operation $*$ satisfying the following axioms.*

- **BG 1**. *B is closed under the operation $*$, that is, $a * b \in B$ for all $a, b \in B$.*
- **BG 2**. *There are identity elements $e_i \in B^{(1)}$, such that $a * e_i = a$ for all $i \in \mathbb{N}$ and for all $a \in B^{(3)}$.*
- **BG 3**. *For each $e_i \in B^{(1)}$, there are $e_j \in B^{(1)}$, such that $e_i * e_j = e_j$ and $e_i \neq e_j$ $(i \neq j)$ for all $i, j \in \mathbb{N}$.*
- **BG 4**. *Each element $a \in B^{(3)}$ has inverse elements $\hat{a}_i \in B^{(2)}$ such that $a * \hat{a}_i = e_i$ for all $i \in \mathbb{N}$.*
- **BG 5**. *The operation $*$ is associative, that is $(a * b) * c = a * (b * c)$ for all $a, b, c \in B$, except if one of elements belongs to $B^{(1)}$ and other to $B^{(2)}$ with different indexes i.e. $a*(e_i*\hat{c}_j) \neq (a*e_i)*\hat{c}_j$.*

Let us identify two principal keystones of β-groups. Firstly by **BG 2** we have infinitely many non trivial identity elements, which enables us create more flexible algebraic structures, secondly, by **BG 4**, every element of $B^{(3)}$ has infinitely many non trivial inverses, i.e. $a * \hat{a}_i = e_i$, $b * \hat{b}_i = e_i$, $c * \hat{c}_i = e_i, \ldots$, for all $i \in \mathbb{N}$.

Exception in **BG 5** protects us from a intricacies, which becomes apparent after Theorem 2.6.

Main question is about consistency of this axiom system. A place where contradictions can occur is **BG 5** conflict with other axioms. Suppose $a, b, c \in B^{(1)}$ and take a look at **BG 5**. Suppose $i \neq j \neq k$, then $(e_i * e_j) * e_k = e_j * e_k = e_k$ and $e_i * (e_j * e_k) = e_i * e_k = e_k$. Other cases in **BG 5**, then elements a, b and c belongs to one of $B^{(j)}$, $j = 1, 2, 3$ subsets easily verifies to be consistent. Except **BG 5**, other axioms that have elements in common and which can lead to contradictions are **BG 2** with **BG 4** and **BG 2** with **BG 3** with **BG 4**. But it is easy to see that these axioms are consistent.

We now prove several propositions that will enable us to manipulate the elements of β-group more easily.

Lemma 2.2. *If $e_i \in B^{(1)}$ are elements of β-group, then $e_i * e_i = e_i$ for all $i \in \mathbb{N}$.*

Proof. By the associativity we have

$$(e_i * e_j) * e_i = e_j * e_i = e_i \quad \text{and} \quad e_i * (e_j * e_i) = e_i * e_i.$$

Hence, by **BG 5** we get $e_i * e_i = e_i$. □

It is easy to see, that by induction we can easily prove generic result $e_{i_1} * e_{i_2} * \ldots * e_{i_n} = e_{i_n}$ for all $i_1, i_2, \ldots, i_n \in \mathbb{N}$.

Lemma 2.3. *For each $a \in B^{(3)}$, and each $e_i \in B^{(1)}$ there is only one $\hat{a}_i \in B^{(2)}$, satisfying equation $a * \hat{a}_i = e_i$ for every fixed $i \in \mathbb{N}$.*

Proof. Suppose that $\hat{a}_i$ and $\hat{a}_j$ $(i \neq j)$ are both inverses satisfying equation $a * \hat{a}_i = e_i = a * \hat{a}_j$ for all $i, j \in \mathbb{N}$. From **BG 3** we know that $e_i \neq e_j$ $(i \neq j)$, and by **BG 4** we have $e_i = a * \hat{a}_i \neq a * \hat{a}_j = e_j$. We get a contradiction. Hence $\hat{a}_i = \hat{a}_j$ and $i = j$. □

Lemma 2.4. *If $\hat{a}_i \in B^{(2)}$, $i \in \mathbb{N}$ are inverse elements of a then $\hat{a}_j \neq \hat{a}_k$ $(j \neq k)$ for every $j, k \in \mathbb{N}$.*

Proof. From the previous lemma we know that for every fixed $i \in \mathbb{N}$ equation $a * \hat{a}_i = e_i$ has only one inverse $\hat{a}_i$ satisfying it. Then the result immediately follows from **BG 3** and **BG 4**. □

Lemma 2.5. *If $a \in B^{(3)}$, $\hat{a}_i \in B^{(2)}$ and $e_i \in B^{(1)}$ are elements of β-group, then for all $i \in \mathbb{N}$,*

$$e_i * \hat{a}_i = \hat{a}_i = \hat{a}_i * e_i.$$

Proof. First, let us investigate an equation

$$e_i * \hat{a}_i = x,$$

then $a * (e_i * \hat{a}_i) = a * x$ and $(a * e_i) * \hat{a}_i = a * x$. That is $a * \hat{a}_i = a * x$ and $e_i = a * x$. Finally from **BG 4** we get $x = \hat{a}_i$.
Secondly, take a look to an equation

$$\hat{a}_i * e_i = x,$$

then $a * (\hat{a}_i * e_i) = a * x$ and $(a * \hat{a}_i) * e_i = a * x$. That is $e_i * e_i = a * x$ and $e_i = a * x$. Finally from **BG 4** we get $x = \hat{a}_i$. Hence, from the first and the second equations follows a statement of the theorem. □

Theorem 2.6. *If $a \in B^{(3)}$, $\hat{a}_i \in B^{(2)}$ and $e_i \in B^{(1)}$ are elements of β-group, then for all $i \in \mathbb{N}$,*

1° $e_i * a = a$;
2° $\hat{a}_i * a = e_i$.

Proof. 1° We have an equation of the form

$$e_i * a = x,$$

then $(e_i * a) * \hat{a}_j = x * \hat{a}_j$ and $e_i * (a * \hat{a}_j) = x * \hat{a}_j$. That is $e_i * e_j = x * \hat{a}_j$ and $e_j = x * \hat{a}_j$. By **BG 4** we have $x = a$.
2° We have an equation of the form

$$\hat{a}_i * a = x,$$

then $(\hat{a}_i * a) * \hat{a}_i = x * \hat{a}_i$ and $\hat{a}_i * (a * \hat{a}_i) = x * \hat{a}_i$. That is $\hat{a}_i * e_i = x * \hat{a}_i$ and $\hat{a}_i = x * \hat{a}_i$. By Lemma 2.5 we get $x = e_i$. □

As a conclusion of the theorem we can rewrite the second and the fourth axioms in the form

BG 2. *There are identity elements $e_i \in B^{(1)}$, such that $a*e_i = a = e_i*a$ for all $i \in \mathbb{N}$ and for all $a \in B^{(3)}$.*
BG 4. *Each element $a \in B^{(3)}$ has inverse elements $\hat{a}_i \in B^{(2)}$ such that $a * \hat{a}_i = e_i = \hat{a}_i * a$ for all $i \in \mathbb{N}$.*

Now, suppose that there is no exception in **BG 5**. Then we can easily prove analogous result to Lemma 2.5, for all $i, j, k \in \mathbb{N}$,

$$e_i * \hat{a}_j = \hat{a}_k * e_j,$$

and the Theorem 2.6.2° then look like $\hat{a}_i * a = e_j$. Suppose that $i \neq j \neq k$, then $(e_i * \hat{a}_j) * a = (\hat{a}_k * e_j) * a$ and $e_i * (\hat{a}_j * a) = \hat{a}_k * (e_j * a)$. That is $e_i * e_m = \hat{a}_k * a$ and $e_m = e_n$. We do not get a contradiction with **BG 3** since we can choose $m = n$ for all $m, n \in \mathbb{N}$. But to avoid ambiguity we choose **BG 5** with an exception.

Now we can examine some basic group properties which can be found in classical algebra or group theory texts Lang (2005) and Rotman (1999) .

Theorem 2.7. *If $a, b \in B^{(3)}$ and $\hat{a}_i, \hat{b}_i \in B^{(2)}$ are elements of β-group, then, for all $i \in \mathbb{N}$*

$$\widehat{(a*b)}_i = \hat{b}_i * \hat{a}_i.$$

Proof. For all $i \in \mathbb{N}$ we have

$$\begin{aligned}(a*b)*(\hat{b}_i * \hat{a}_i) &= a*((b*\hat{b}_i)*\hat{a}_i)\\ &= a*(e_i*\hat{a}_i) = a*\hat{a}_i = e_i.\end{aligned}$$

Hence $\hat{b}_i * \hat{a}_i$ are inverses of $a*b$. □

Theorem 2.8. *If $a, b, c \in B^{(3)}$ are elements of β-group, then $a*b = a*c$ or $b*a = c*a$ implies $b = c$.*

Proof. Suppose $a*b = a*c$. Then, for all $i \in \mathbb{N}$, $\hat{a}_i*(a*b) = \hat{a}_i*(a*c)$ and $(\hat{a}_i*a)*b = (\hat{a}_i*a)*c$. That is, by the Theorem 2.6.2°, $e_i*b = e_i*c$ and $b = c$.
Similarly, suppose $b*a = c*a$. Then, for all $i \in \mathbb{N}$, $(b*a)*\hat{a}_i = (c*a)*\hat{a}_i$ and $b*(a*\hat{a}_i) = c*(a*\hat{a}_i)$. That is, $b*e_i = c*e_i$ and $b = c$. □

Now we can ask about solution of the equation $x*a = b$. How many solutions this equation has? First of all let us look at the simple case $a, b \in B^{(1)}$, then $x*e_i = e_i$ by **BG 3**. And by the same axiom we get $x = e_j$ for any $i, j \in \mathbb{N}$. Hence, as $a, b \in B^{(1)}$, then $x \in B^{(1)}$ and we have infinitely many solutions. But if we look at the equation $e_j*x = e_i$, then it is easy to see that it has unique solution for every fixed $i \in \mathbb{N}$.

If we take $a, b \in B^{(3)}$, then $(x*a)*\hat{a}_i = b*\hat{a}_i$ and $x*e_i = b*\hat{a}_i$. In this case we need more information about elements $\hat{a}_i$ and b to tell something about x.

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